

# Black hole Skyrme in a generalized Skyrme model

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**ABSTRACT:** We study a Skyrme-like model with the Skyrme term and a sixth-order derivative term as higher-order terms, coupled to gravity and we construct Schwarzschild black hole Skyrme hair. We find, surprisingly, that the sixth-order derivative term alone cannot stabilize the black hole hair solutions; the Skyrme term with a large enough coefficient is a necessity.

**KEYWORDS:** Black Holes, Solitons Monopoles and Instantons

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## 1 Introduction

Black holes are generally believed to be characterized by two properties at asymptotically far distances, namely their masses and their global charges. This is known as the weak no-hair conjecture. The first stable counter example to this no-hair conjecture was made in the framework of the Skyrme model, i.e. a black hole with Skyrme hair [1–7] (for a review, see ref. [8]). The Skyrme model is a scalar field theory based on the chiral Lagrangian with the addition of a quartic derivative term, which is like a curvature term on the internal (target) space of the model [9, 10]. In flat space the Skymion is a map from space ( $\mathbb{R}^3$ ) to an  $SU(2)$  target space, which is characterized by  $\pi_3(S^3)$ , giving rise to topological solitons; i.e. the Skymions. As implied by the fact that they are topological, their charges — called baryon charges — are integers in flat space. The Skymion with a black hole is interpreted as a black hole with scalar hair and the asymptotic behavior of the Skymion is very similar to that in flat space. Near the black hole the Skymion is deformed, nevertheless. Now when the black hole is formed with the Skymion surrounding it, the Skymion loses a fraction of its charge; this happens due to the fact that the profile function of the Skymion, which normally “winds”  $\pi$  to complete the 3-cycle, only “winds”  $\pi - \epsilon$ . When the black hole horizon — and hence its mass — becomes larger than a certain critical value then the Skymion ceases to exist and the Skyrme hair becomes unstable.

Another twist to the Skymion solution in flat space is that when it is the hair of a black hole, two branches of solutions (fixed points of the action) open up [1]; one of these two branches of solutions contains, however, unstable Skymion solutions. The two branches bifurcate at the above mentioned critical mass or horizon radius, beyond which no stable solution exists. If we pick a point on the stable branch and take the limit of the black hole mass going to zero, then the solutions converge smoothly to that of the flat space. If we now pick a point on the unstable branch, the answer depends on whether the gravitational coupling is turned on or not; if it is turned on — which is tantamount to the gravitational backreaction being taken into account — then the conclusion remains the same; the solution converges to that of flat space. If the gravitational coupling is turned off, however, then the solution becomes discontinuous and ceases to exist — the limit is hence not well defined.

Apart from the seminal result of Luckock et al. (ref. [1]), and the papers that followed; other variants of the Skyrme black hole hair system have been studied in the literature. The most natural generalization is to turn on a nonvanishing cosmological constant; in refs. [11, 12] and [13] the black hole Skyrme hair was ported to anti-de Sitter and de Sitter spacetimes, respectively. The late-time evolution of the radiation emitted from the black hole with Skyrme hair was studied in refs. [14, 15]. Gravitating sphalerons in the Einstein-Skyrme model have been constructed in ref. [16]. Quantization of collective coordinates in the Skyrminion black hole was carried out [17]. Another natural generalization of the black hole Skyrminion system, is to consider the Skyrminion surrounding the black hole to have a higher charge (winding); a particular class of axially symmetric solutions has been found in refs. [18, 19] and quantization of collective coordinates was considered in such systems as well [20]. Recently, it has been contemplated that black holes do not necessarily violate the baryon number when the possibility of black hole Skyrme hair is taken into consideration [21].

An interesting question is: what is the foundation of the stabilizing mechanism of the black hole hair? If we turn off the Skyrme term, the scalar hair is not stable. In light of recent developments in the Skyrme model, which was motivated by a completely different effect — namely the large binding energy of the multi-Skyrmion, being too large for the Skyrmions to be interpreted as nuclei — a sixth-order derivative term has been introduced [22, 23] and this model has been dubbed the BPS-Skyrme model (neutron stars have been studied in the framework of the BPS-Skyrme model [24, 25] and we have recently found gravitating analytic and numerical Skyrminion solutions in the BPS-Skyrme model [26]). For the story of the binding energy, the sextic term has the interesting property that a saturable BPS bound exists in the subset of the model containing only said sextic term as well as a potential term. Using Derrick’s theorem [27], any higher-order derivative term can stabilize the Skyrminion solution in flat space, by balancing the pressure with respect to that of the kinetic (Dirichlet) term and/or the potential. As for the hair of the black hole, however, it is far less trivial which kind of terms can stabilize the black hole hair. One may naively think that we can substitute the Skyrme term with the sixth-order derivative term and retain a similar black hole hair solution. Our findings, however, suggest otherwise. Although we can add the sextic term to the model and have stable black hole hair; the Skyrme term with a positive coefficient is a necessity. This is the main result of our paper.

Another result found in this paper concerns the unstable branches mentioned above. When the gravitational coupling is turned on in the Skyrme model without the sextic term — corresponding to taking gravitational backreaction into account — then the unstable branches of solutions smoothly converge back to the flat space Skyrminion solution in the limit of vanishing black hole size. Once we turn on the sextic term (sixth-order derivative term), there is a small, but finite, critical value for the coefficient of said term, for which the unstable branches end at a finite horizon radius: hence the limit is not smooth.

The paper is organized as follows. Section 2 introduces the model and the governing equations for the black hole in a Schwarzschild metric with Skyrme(-like) hair. Section 3 presents the numerical results and finally section 4 concludes with a discussion.

## 2 The model

The model is a nonlinear sigma model of Skyrme-type with higher-derivative terms up to sixth order, coupled to gravity and the action reads

$$S = \int d^4x \sqrt{-g} \mathcal{L}, \quad \mathcal{L} = \mathcal{L}_S + \mathcal{L}_G, \quad (2.1)$$

$$\mathcal{L}_S = c_2 \mathcal{L}_2 + c_4 \mathcal{L}_4 + c_6 \mathcal{L}_6 - \delta c_0 V(U), \quad \mathcal{L}_G = \frac{1}{16\pi G} R, \quad (2.2)$$

where the  $n$ -th order Lagrangians are given by

$$\mathcal{L}_2 = -\frac{1}{4} g^{\mu\mu'} \text{tr}(L_\mu L_{\mu'}), \quad (2.3)$$

$$\mathcal{L}_4 = \frac{1}{32} g^{\mu\mu'} g^{\nu\nu'} \text{tr}([L_\mu, L_\nu][L_{\mu'}, L_{\nu'}]), \quad (2.4)$$

$$\mathcal{L}_6 = -\frac{1}{144} g_{\mu\mu'} (-g^{-1}) (\epsilon^{\mu\nu\rho\sigma} \text{tr}[L_\nu L_\rho L_\sigma]) (\epsilon^{\mu'\nu'\rho'\sigma'} \text{tr}[L_{\nu'} L_{\rho'} L_{\sigma'}]), \quad (2.5)$$

where  $L_\mu \equiv U^\dagger \partial_\mu U$  is the left-invariant current,  $U = \sigma \mathbf{1}_2 + i\pi^a \tau^a$ ,  $a = 1, 2, 3$  is the Skyrme field with the constraint  $\det U = 1$ ,  $g$  is the determinant of the metric and we are using the mostly-negative signature of the metric.  $\mathcal{L}_2$  is the standard kinetic term,  $\mathcal{L}_4$  is the Skyrme term and  $\mathcal{L}_6$  is the baryon current density squared, which is inspired by the BPS Skyrme model [22].

In the remainder of the paper, we will use the terminology

$$2 + 4 \text{ model} : \quad c_4 > 0, \quad c_6 = 0, \quad (2.6)$$

$$2 + 4 + 6 \text{ model} : \quad c_4 > 0, \quad c_6 > 0. \quad (2.7)$$

The symmetry of  $\mathcal{L}_S$  for  $V = 0$  is  $\tilde{G} = \text{SU}(2)_L \times \text{SU}(2)_R$  acting on  $U$  as  $U \rightarrow U' = g_L U g_R^\dagger$  and thus  $L_\mu$  is manifestly covariant. Finite energy configurations require that  $U$  asymptotically takes on a constant value, e.g.  $U = \mathbf{1}_2$ . Hence in the vacuum  $\tilde{G}$  is spontaneously broken down to  $\tilde{H} \simeq \text{SU}(2)_{L+R}$ , which in turn acts on  $U$  as  $U \rightarrow U' = g U g^\dagger$ . The target space is therefore  $\tilde{G}/\tilde{H} \simeq \text{SU}(2)_{L-R}$ .

For concreteness we will use the potential

$$V(U) = \frac{1}{16} \text{tr} \left[ (2\mathbf{1}_2 - U - U^\dagger)(2\mathbf{1}_2 + U + U^\dagger) \right], \quad (2.8)$$

which is sometimes called the modified pion mass term [28–31], see also refs. [32–37]. This potential breaks  $\tilde{G}$  to  $\text{SU}(2)_{L+R}$  explicitly.

The sixth-order term is written in a way where it is manifest that it is the baryon current squared. It is however slightly easier to work with the term after rewriting it in the following form [38]

$$\mathcal{L}_6 = \frac{1}{96} g^{\mu\mu'} g^{\nu\nu'} g^{\rho\rho'} \text{tr}(L_\mu [L_\nu, L_\rho]) \text{tr}(L_{\mu'} [L_{\nu'}, L_{\rho'}]). \quad (2.9)$$

In this paper we will consider the Schwarzschild metric

$$ds^2 = N^2(r) C(r) dt^2 - \frac{1}{C(r)} dr^2 - r^2 d\theta^2 - r^2 \sin^2 \theta d\phi^2, \quad (2.10)$$

with

$$C(r) = 1 - \frac{2m(r)}{r}, \quad (2.11)$$

which is appropriate for studying a single Skyrmion — which is also spherically symmetric — for which we choose the so-called hedgehog Ansatz

$$U = \cos f(r) + i\hat{x}^a \tau^a \sin f(r), \quad (2.12)$$

where  $\hat{x}$  is a spatial unit vector and  $a = 1, 2, 3$ . Using the hedgehog Ansatz, we can write the static mass of the Skyrmion as

$$M = 4\pi \int_{r_h}^{\infty} dr r^2 N \left[ c_2 \left( \frac{1}{2} C f_r^2 + \frac{\sin^2 f}{r^2} \right) + c_4 \frac{\sin^2 f}{r^2} \left( C f_r^2 + \frac{\sin^2 f}{2r^2} \right) + c_6 C \frac{\sin^4(f) f_r^2}{r^4} + \frac{\delta c_0}{2} \sin^2 f \right], \quad (2.13)$$

where  $f_r \equiv \partial_r f$  and  $r_h$  is the horizon radius. The mass of the Skyrmion is the energy density integrated from the horizon to infinity. We will now change variables to the dimensionless coordinate  $\rho \equiv \sqrt{\frac{c_0}{c_2}} r$  and rescale the coefficients  $c_4 \rightarrow \frac{c_2^2}{c_0} c_4$  and  $c_6 \rightarrow \frac{c_2^3}{c_0^3} c_6$ . Although we have rescaled the coordinates by the coefficient of the mass term, we insert a dimensionless mass,  $\delta$  which can take the value 0 or 1. If  $\delta = 0$  then  $c_0$  is still the unit of the would-be mass ( $c_0$  can never vanish). In the case of  $\delta = 0$ ,  $c_0$  can be adjusted such that  $c_4 = 1$ . We can now write the mass as follows

$$M = 4\pi \sqrt{\frac{c_2^3}{c_0}} \int_{\rho_h}^{\infty} d\rho \rho^2 N \left[ \left( \frac{1}{2} C f_\rho^2 + \frac{\sin^2 f}{\rho^2} \right) + c_4 \frac{\sin^2 f}{\rho^2} \left( C f_\rho^2 + \frac{\sin^2 f}{2\rho^2} \right) + c_6 C \frac{\sin^4(f) f_\rho^2}{\rho^4} + \frac{\delta}{2} \sin^2 f \right], \quad (2.14)$$

where the dimensionless horizon radius is  $\rho_h = \sqrt{\frac{c_0}{c_2}} r_h$  and we define

$$\mu(\rho) = \sqrt{\frac{c_0}{c_2}} m(r). \quad (2.15)$$

$c_4$ ,  $c_6$  and  $\delta = 0, 1$  are now dimensionless parameters.

The baryon current is

$$\mathcal{B}^\mu = -\frac{1}{24\pi^2} \frac{\epsilon^{\mu\nu\rho\sigma}}{\sqrt{-g}} \text{tr}(L_\nu L_\rho L_\sigma), \quad (2.16)$$

and integrating the time component of this gives the baryon charge

$$B = \int d^3x \sqrt{-g} \mathcal{B}^0 = -\frac{2}{\pi} \int_{\rho_h}^{\infty} d\rho \sin^2(f) f_\rho = \frac{2f(\rho_h) - \sin 2f(\rho_h)}{2\pi}, \quad (2.17)$$

and thus the total baryon charge  $B$  is less than unity for any  $f(\rho_h) < \pi$  (we have used the asymptotic boundary condition  $f(\infty) = 0$ ).

The equation of motion for the Skyrme field profile  $f$  is given by

$$\begin{aligned} & C \left( \rho^2 + 2c_4 \sin^2 f + \frac{2c_6 \sin^4 f}{\rho^2} \right) f_{\rho\rho} \\ & + \left[ \left( C_\rho + C \frac{N_\rho}{N} \right) \left( \rho^2 + 2c_4 \sin^2 f + \frac{2c_6 \sin^4 f}{\rho^2} \right) + C \left( 2\rho - \frac{4c_6 \sin^4 f}{\rho^3} \right) \right] f_\rho \\ & + C \sin(2f) \left( c_4 + \frac{2c_6 \sin^2 f}{\rho^2} \right) f_\rho^2 - \sin(2f) \left( 1 + \frac{c_4 \sin^2 f}{\rho^2} + \frac{\delta \rho^2}{2} \right) = 0. \end{aligned} \quad (2.18)$$

The energy-momentum tensor can readily be calculated as

$$\begin{aligned} T_{\mu\nu} = & -\frac{1}{2} \text{tr} (L_\mu L_\nu) + \frac{c_4}{8} g^{\rho\sigma} \text{tr} ([L_\mu, L_\rho][L_\nu, L_\sigma]) + \frac{c_6}{16} g^{\rho\sigma} g^{\lambda\omega} \text{tr} (L_\mu [L_\rho, L_\lambda]) \text{tr} (L_\nu [L_\sigma, L_\omega]) \\ & - g_{\mu\nu} \mathcal{L}_S. \end{aligned} \quad (2.19)$$

Writing out the nonzero components, we have

$$\frac{1}{c_0} T_{tt} = CN^2 \left[ \frac{1}{2} C f_\rho^2 + \frac{\sin^2 f}{\rho^2} + c_4 \frac{\sin^2 f}{\rho^2} \left( C f_\rho^2 + \frac{\sin^2 f}{2\rho^2} \right) + c_6 C \frac{\sin^4(f) f_\rho^2}{\rho^4} + \frac{\delta}{2} \sin^2 f \right], \quad (2.20)$$

$$\frac{1}{c_0} T_{\rho\rho} = \frac{1}{2} f_\rho^2 - \frac{\sin^2 f}{C\rho^2} + c_4 \frac{\sin^2 f}{\rho^2} \left( f_\rho^2 - \frac{\sin^2 f}{2C\rho^2} \right) + c_6 \frac{\sin^4(f) f_\rho^2}{\rho^4} - \delta \frac{\sin^2 f}{2C}, \quad (2.21)$$

$$\frac{1}{c_0} T_{\theta\theta} = \frac{1}{c_0} \frac{T_{\phi\phi}}{\sin^2 \theta} = -\frac{1}{2} \rho^2 C f_\rho^2 + c_4 \frac{\sin^4 f}{2\rho^2} + c_6 C \frac{\sin^4(f) f_\rho^2}{\rho^2} - \frac{\delta}{2} \rho^2 \sin^2 f. \quad (2.22)$$

We are now ready to obtain the Einstein equations

$$G_{\mu\nu} = 8\pi G T_{\mu\nu}, \quad (2.23)$$

and defining  $\alpha \equiv 8\pi G c_0$ , we can write down the resulting equations by taking suitable linear combinations

$$\frac{1}{\alpha} \frac{N_\rho}{N} = \frac{1}{2} \rho f_\rho^2 + c_4 \frac{\sin^2(f) f_\rho^2}{\rho} + c_6 \frac{\sin^4(f) f_\rho^2}{\rho^3}, \quad (2.24)$$

$$\begin{aligned} \frac{1}{\alpha} C_\rho = & \frac{1-C}{\alpha\rho} - C \left( \frac{1}{2} \rho f_\rho^2 + \frac{c_4 \sin^2(f) f_\rho^2}{\rho} + \frac{c_6 \sin^4(f) f_\rho^2}{\rho^3} \right) - \frac{\sin^2 f}{\rho} - \frac{c_4 \sin^4 f}{2\rho^3} - \frac{\delta}{2} \rho \sin^2 f. \end{aligned} \quad (2.25)$$

We can eliminate the field,  $N$ , by inserting eq. (2.24) into eq. (2.18) and simplify the coefficient of  $f_\rho$  by using eq. (2.25). The resulting system of equations is then given by

$$\begin{aligned} & C \left( \rho^2 + 2c_4 \sin^2 f + \frac{2c_6 \sin^4 f}{\rho^2} \right) f_{\rho\rho} \\ & + \left[ \left( 1 - \alpha \sin^2 f - \frac{\alpha c_4 \sin^4 f}{2\rho^2} - \frac{1}{2} \alpha \delta \rho^2 \right) \left( \rho + \frac{2c_4 \sin^2 f}{\rho} + \frac{2c_6 \sin^4 f}{\rho^3} \right) \right. \\ & \quad \left. + C \left( \rho - \frac{2c_4 \sin^2 f}{\rho} - \frac{6c_6 \sin^4 f}{\rho^3} \right) \right] f_\rho \\ & + C \sin(2f) \left( c_4 + \frac{2c_6 \sin^2 f}{\rho^2} \right) f_\rho^2 - \sin(2f) \left( 1 + \frac{c_4 \sin^2 f}{\rho^2} + \frac{\delta \rho^2}{2} \right) = 0, \end{aligned} \quad (2.26)$$

and eq. (2.25).

In order to find numerical solutions to the system of equations (2.26) and (2.25), it will be convenient to use a shooting method for ordinary differential equations (ODEs). For that we need boundary conditions at the horizon (at  $\rho_h$ ) with a shooting parameter as well as boundary conditions at infinity. The boundary condition at the horizon is

$$\lim_{\rho \rightarrow \rho_h} C = 1 - \frac{2\mu(\rho_h)}{\rho_h} = 0, \quad (2.27)$$

and hence  $\mu(\rho_h) = \rho_h/2$ .  $f(\rho_h) = f_h$  is the shooting parameter and by taking the limit  $\rho \rightarrow \rho_h$  of eq. (2.26), we get

$$C_\rho(\rho_h) \left( \rho_h^2 + 2c_4 \sin^2 f_h + \frac{2c_6 \sin^4 f_h}{\rho_h^2} \right) f_\rho(\rho_h) - \sin(2f_h) \left( 1 + \frac{c_4 \sin^2 f_h}{\rho_h^2} + \frac{\delta \rho_h^2}{2} \right) = 0. \quad (2.28)$$

Now we need to evaluate  $C_\rho$  at the horizon

$$\lim_{\rho \rightarrow \rho_h} C_\rho = -\frac{2\mu_\rho(\rho_h)}{\rho_h} + \frac{1}{\rho_h} = -\alpha \left( \frac{\sin^2 f_h}{\rho_h} + \frac{c_4 \sin^4 f_h}{2\rho_h^3} + \frac{\delta}{2} \rho_h \sin^2 f_h \right) + \frac{1}{\rho_h}, \quad (2.29)$$

which follows straightforwardly from eq. (2.25).

Summarizing, we have the boundary conditions at the horizon

$$f(\rho_h) = f_h, \quad (2.30)$$

$$f_\rho(\rho_h) = \frac{\rho_h^3 \sin(2f_h) (2\rho_h^2 + 2c_4 \sin^2 f_h + \delta \rho_h^4)}{[2\rho_h^2 - \alpha \sin^2 f_h (2\rho_h^2 + c_4 \sin^2 f_h + \delta \rho_h^4)] (\rho_h^4 + 2c_4 \rho_h^2 \sin^2 f_h + 2c_6 \sin^4 f_h)}, \quad (2.31)$$

$$\mu(\rho_h) = \frac{\rho_h}{2}, \quad (2.32)$$

while at infinity they are

$$f(\infty) = 0, \quad \mu_\rho(\infty) = 0, \quad (2.33)$$

where the second condition follows from the first and corresponds to

$$\lim_{\rho \rightarrow \infty} C = 1 - \frac{\text{const}}{\rho}, \quad (2.34)$$

i.e., the metric is asymptotically Schwarzschild. In total there are exactly three boundary conditions on our system (which is what is necessary) and one shooting parameter.

Note that the first derivative of the Skymion profile function,  $f_\rho$ , is negative at the horizon (as it should be), only when

$$\Xi \equiv 2\rho_h^2 - \alpha \sin^2 f_h (2\rho_h^2 + c_4 \sin^2 f_h + \delta \rho_h^4), \quad (2.35)$$

is positive, because  $\sin 2f_h$  is negative for  $f_h \in (\frac{1}{2}\pi, \pi)$ , which is the relevant range of the shooting parameter. When  $\Xi$  vanishes, the first derivative is not defined on the horizon and solutions cease to exist. Notice that  $\Xi = 0$  corresponds to a vanishing Hawking temperature, since

$$T_H = \frac{N(\rho_h)C_\rho(\rho_h)}{4\pi}, \quad (2.36)$$

and  $\Xi = 2\rho_h^3 C_\rho(\rho_h) = 0$  is equivalent to  $C_\rho(\rho_h) = 0$  for  $\rho_h > 0$ .

### 3 Numerical solutions

We will now pursue finding numerical solutions to the black hole Skyrme system. Eq. (2.28) implies that the coefficient of  $f_{\rho\rho}$  vanishes at the horizon, which is problematic for a shooting algorithm, as we would like to make a dynamic system of equations as

$$\partial_\rho \begin{pmatrix} f \\ f_\rho \\ \mu \end{pmatrix} = M \begin{pmatrix} f \\ f_\rho \\ \mu \end{pmatrix}, \quad (3.1)$$

where  $M$  is some matrix (functional); the second row of the right-hand side is defined by eq. (2.26) and the last row by eq. (2.25). However the right-hand side of the second row is not well defined at the horizon (the coefficient of  $f_{\rho\rho}$  in eq. (2.26) vanishes at the horizon). Therefore we start the shooting from a very small radius  $\rho_\epsilon$  and calculate the values of the fields at  $\rho_h + \rho_\epsilon$  as

$$f(\rho_h + \rho_\epsilon) = f_h + \rho_\epsilon f_\rho(\rho_h), \quad (3.2)$$

$$f_\rho(\rho_h + \rho_\epsilon) = f_\rho(\rho_h), \quad (3.3)$$

$$\mu(\rho_h + \rho_\epsilon) = \frac{\rho_h}{2} + \rho_\epsilon \mu_\rho(\rho_h), \quad (3.4)$$

where  $\mu_\rho(\rho_h)$  is given by eq. (2.29). This is a good approximation if  $\rho_\epsilon$  is extremely small. In the numerical calculations we have found that  $\rho_\epsilon \lesssim 10^{-5}$  is small enough for allowing for the linear approximation and large enough to start the shooting algorithm.

From  $\rho_h + \rho_\epsilon$  we employ a standard fourth-order Runge-Kutta method to integrate the equations (2.25)–(2.26) up to an appropriately chosen cutoff.

We are now ready to present the numerical results. We start by reproducing the well-known results in the 2+4 model, which is simply the standard Skyrme model with the addition of the modified pion mass. After rescaling we have two free parameters: the gravitational coupling  $\alpha$  and  $c_4$ . Actually, if we were to consider the model without the potential term, then the rescaling would eliminate  $c_4$  instead of the potential parameter (pion mass).<sup>1</sup>

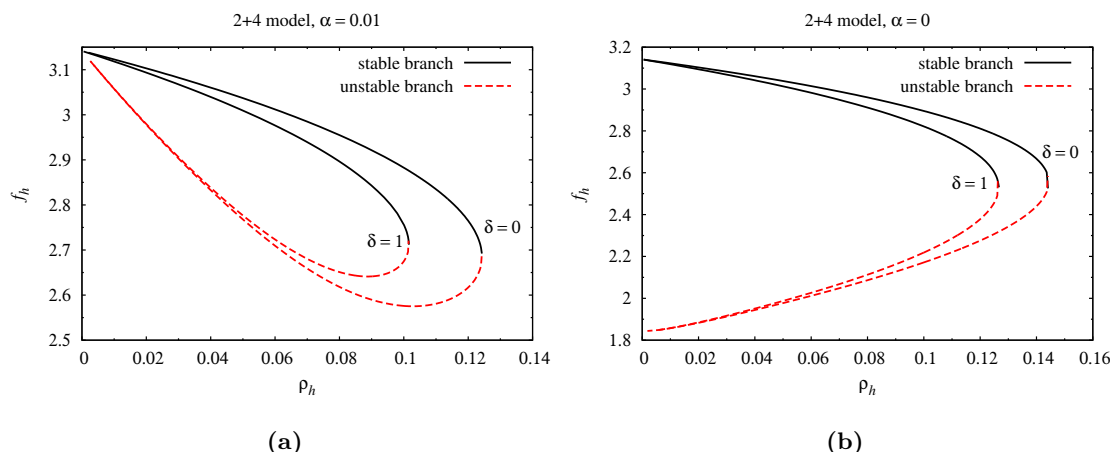
In figure 1 we show the stable and unstable branches in black solid lines and dashed red lines, respectively, and for all combinations of vanishing/nonvanishing gravitational coupling and vanishing/nonvanishing potential parameter  $\delta$  (recall that  $\delta$  after rescaling, can only take the values 0 or 1). The first thing we note is, as explained in the introduction, that with the gravitational coupling turned on, the unstable branch smoothly converges back towards the flat space Skyrme solution as the horizon radius  $\rho_h$  is sent to zero; whereas in the case of vanishing gravitational coupling it does not. The unstable branch continues down in the  $f_h$  direction and in the limit of vanishing  $\rho_h$  the solution is discontinuous and ceases to exist [1].

We now turn on a positive value for the coefficient of the sextic term,  $c_6 > 0$ ; this corresponds to the 2+4+6 model. A priori one would not expect substantial differences with respect to the 2+4 model discussed above, because in flat space the soliton solutions (in this

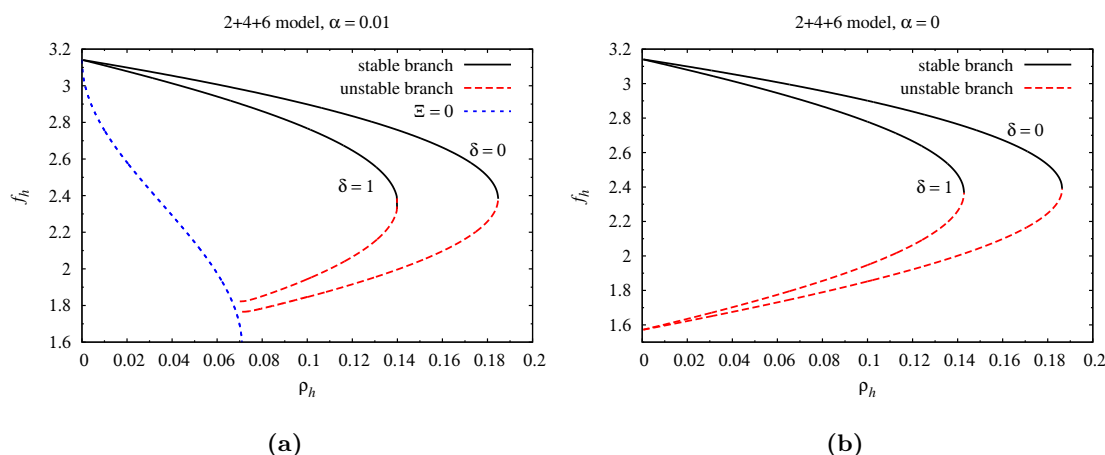
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<sup>1</sup>Of course this mass parameter is not related to the mass of the physical pion in QCD.



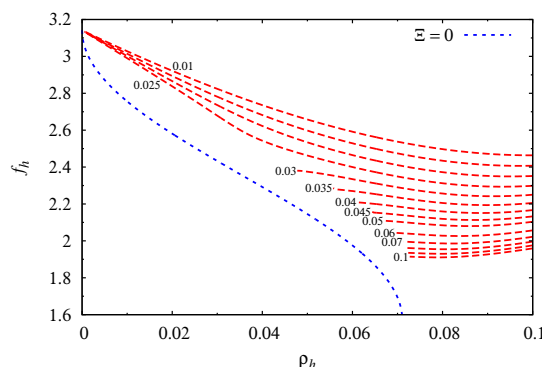


**Figure 1.** Branches of solutions for the 2+4 model with and without mass term ( $\delta = 0, 1$ ) for (a) gravitational coupling  $\alpha = 0.01$  and (b)  $\alpha = 0$ . In this figure  $c_4 = 1$ .



**Figure 2.** Branches of solutions for the 2+4+6 model with and without mass term ( $\delta = 0, 1$ ) for (a) gravitational coupling  $\alpha = 0.01$  and (b)  $\alpha = 0$ . The blue dashed line to the left in figure (a) represents the vanishing of  $\Xi = 0$  of eq. (2.35) which corresponds to a vanishing Hawking temperature, at which there is no black hole hair solution. In this figure  $c_4 = c_6 = 1$ .

parameter range; that is, when  $c_4$  is of the same order of magnitude as  $c_6$  or larger) are quite similar [36, 37, 39]. However, by inspection of figure 2 we see that differences in the unstable branches, with respect to the 2+4 model, emerged. The stable branches are quite similar to that of the 2+4 model, except that they are longer; i.e. the Skyrme hair solutions in the 2+4+6 model are stable for much larger black holes than the 2+4 model. The unstable branches without the gravitational coupling turned on remain similar to those of the 2+4 model; however, when the gravitational backreaction is turned on, they do not converge back towards the flat space Skyrme solution. In some sense the unstable branches are on the same trajectory (downwards in the  $(\rho_h, f_h)$  phase diagram) as those without gravi-



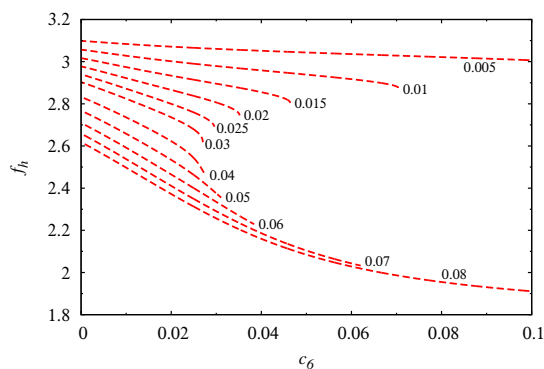
**Figure 3.** Unstable branches of solutions for the 2+4+6 model without mass term for various values of  $c_6 = 0.01, 0.015, 0.02, 0.025, 0.03, 0.035, 0.04, 0.045, 0.05, 0.06, 0.07, 0.08, 0.09, 0.1$ . The values of  $c_6$  are indicated on the figure. The blue dashed line shows where  $\Xi = 0$ . In this figure  $c_4 = 1$ ,  $\delta = 0$  and the gravitational coupling is  $\alpha = 0.01$ .

tational backreaction for decreasing horizon radius,  $\rho_h$ . However, long before reaching the limit of vanishing black hole size, the unstable branches come close to the  $\Xi = 0$  line in the diagram, where  $\Xi$  is defined in eq. (2.35). This phenomenon happens both with and without the potential term. The  $\Xi = 0$  line is defined by the position of the pole in the first radial derivative of the Skyrme profile function  $f_\rho(\rho_h)$  at the horizon radius, and corresponds to a vanishing Hawking temperature. It is intuitively clear that when the Skyrme profile blows up at the black hole horizon, no continuous stable black hole hair solution exists. It is also clear that if the Hawking temperature vanishes for a finite black hole mass, then the entropy would have to blow up; this should not happen in a physical system and hence it signals an instability of the black hole hair. We observe from figure 2 that the unstable branch of solutions ceases to exist slightly before  $\Xi = 0$ , but quite close to this line in the diagram.

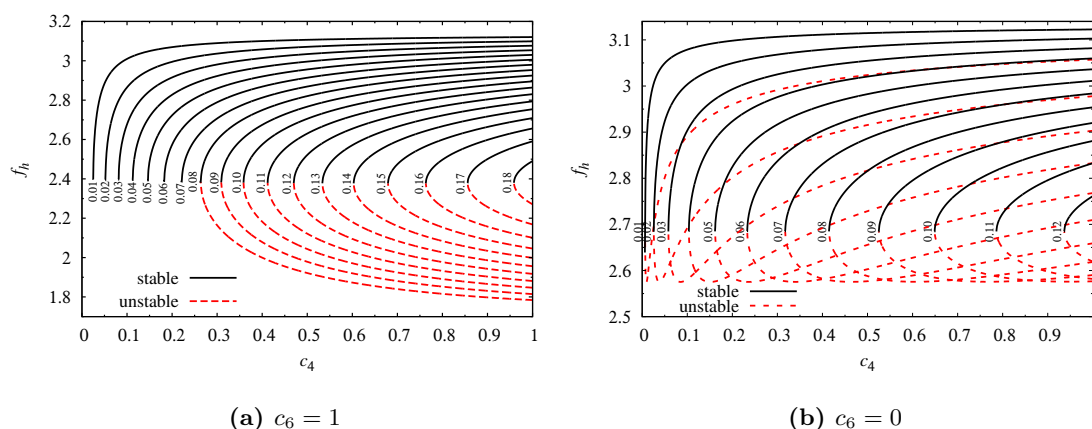
As we know that in the 2+4 model, the unstable branch moves upwards in the  $(\rho_h, f_h)$  phase diagram for decreasing horizon radius  $\rho_h$ , there should be some critical value of  $c_6$  for which the unstable branches start to end at a finite horizon radius  $\rho_h > 0$ . We therefore consider taking the limit of  $c_6 \rightarrow 0$  and see when the unstable branches start to exist in the limit of  $\rho_h \rightarrow 0$ . Figure 3 shows the phase diagram with only the unstable branches for various values of the coefficient of the sextic term,  $c_6$ , and we can see from the figure that for  $c_4 = 1, \delta = 0$ , the critical value of  $c_6$  is between 0.025 and 0.03.

In figure 4 we show the same physics, but in terms of  $c_6$  and  $f_h$ ; the different curves depict various horizon radii,  $\rho_h$ . This figure clearly shows that if we turn off the sextic term,  $c_6 = 0$ , then all (the shown) horizon radii have solutions. Curiously, the lines open up in the limit of  $\rho_h \rightarrow 0$  and allow for bigger  $c_6$  than for example  $\rho_h = 0.03$ , which is the most restricting radius in the diagram. From figure 3 we estimated that the critical value of  $c_6$  for which the unstable branch ends at a finite horizon radius,  $\rho_h > 0$ , is about 0.025-0.03; while from figure 4, we can confirm that it is slightly less than 0.03, so in accord with the previous estimate.

Our final numerical investigation considers turning off the Skyrme term for fixed coefficient of the sextic term  $c_6 = 1$ . Figure 5a shows both stable and unstable solutions in the



**Figure 4.** Families of unstable solutions for the 2+4+6 model for various horizon radii,  $\rho_h = 0.005, 0.01, 0.015, 0.02, 0.025, 0.03, 0.04, 0.05, 0.06, 0.07, 0.08$  as functions of  $c_6$ . The horizon radii are indicated on the figure. In this figure  $c_4 = 1$ ,  $\delta = 0$  and the gravitational coupling is  $\alpha = 0.01$ .



**Figure 5.** Families of solutions for (a) the 2+4+6 model and (b) the 2+4 model, for various horizon radii, (a)  $\rho_h = 0.01, 0.02, 0.03, 0.04, 0.05, 0.06, 0.07, 0.08, 0.09, 0.10, 0.11, 0.12, 0.13, 0.14, 0.15, 0.16, 0.17, 0.18$  and (b)  $\rho_h = 0.01, 0.02, 0.03, 0.04, 0.05, 0.06, 0.07, 0.08, 0.09, 0.10, 0.11, 0.12$  as functions of  $c_4$ . The horizon radii are indicated on the figure. We can see from the figure that for decreasing values of  $c_4$ , only smaller and smaller horizon radii exist — even on the stable branch. In this figure  $\delta = 0$  and the gravitational coupling is  $\alpha = 0.01$ .

$(c_4, f_h)$  diagram for various horizon radii  $\rho_h$ . For comparison, we show also the analogous figure  $(c_4, f_h)$  for  $c_6 = 0$  in figure 5b where it is clear that the black hole Skyrme hair will cease to exist when the Skyrme term is turned off. The biggest difference between the two panels lies in the unstable branches, because in the 2+4 model the unstable branches return to the flat-space Skyrme when the black hole horizon radius is sent to zero ( $\rho_h \rightarrow 0$ ). Fixing  $c_4$  we can read off the  $f_h$  branch as function of the horizon radii by looking at a vertical line in the figure. Following a fixed horizon radius,  $\rho_h$ , we can see at which value of the Skyrme term coefficient,  $c_4$ , the solutions cease to exist. Interestingly — and this is one of the main findings of this paper — *all* solutions cease to exist in the limit of  $c_4 \rightarrow 0$ , even though we have the sextic term turned on. We can also see that the unstable branches cease to exist

quite before the stable branches (in the case of  $c_6 = 1$ , see figure 5a). We can physically understand that the bifurcation point — which is the maximal size of black hole that can support the Skyrme hair — simply goes to zero in the limit of  $c_4 \rightarrow 0$  for fixed  $c_6 = 1$ ; we can perhaps say that the black hole eats the Skyrme hair if there is no Skyrme term turned on.

## 4 Discussion and conclusion

In this paper we have considered Schwarzschild black holes with Skyrme hair in a Skyrme-like model with the addition of a sixth-order derivative term as well as a potential term. We first reproduce the expected branches of solutions in the  $(\rho_h, f_h)$  phase diagram for the Skyrme model with the sextic term turned off. Then turning on the sextic term, we find that the unstable branches are modified and end at finite horizon radii — beyond which no unstable solution exists, for all but very small values of the coefficient of the sextic term. This is due to the unstable branches coming too close to a line in the phase diagram ( $\Xi = 0$ ) where the Hawking temperature vanishes. Furthermore, we find the quite surprising result that there is no stable or unstable Skyrme hair for any black holes — with or without the potential — in the limit of vanishing coefficient of the Skyrme term,  $c_4 \rightarrow 0$ . This unexpected result implies that, although higher-derivative terms can stabilize Skyrmions in flat space, the sixth-order derivative term cannot stabilize the Schwarzschild black hole hair.

In ref. [26] we observed that there are no black holes in the BPS-Skyrme submodel; i.e. in the case without a kinetic term and without the Skyrme term. This observation was made in the case of a particular potential and so it was not clear whether Skyrme hair solutions in the 2+6 model would be stable or not. Now we have the answer in the negatory. This interesting result begs for the question: under what circumstances does black hole hair exist? It is quite unlikely that any potential of any type will alter this conclusion as potentials tend to collapse the Skyrmions and the black hole already does that without the Skyrme term present — even with the sixth-order derivative term. Let us note that for small values of the Skyrme term coefficient (small  $c_4$ ), the unstable branches come too close to the line in the phase diagram where the Hawking temperature vanishes. Furthermore, we observe that the critical point moves to smaller and smaller black hole horizons for decreasing values of  $c_4$  and eventually leads to a vanishing black hole size even for the stable branches. This in turn means that the *stable branch* is approaching the line where the Hawking temperature is vanishing. Let us also note that the sixth-order derivative term itself — without the presence of the second-order kinetic term — leads to a theory described by a perfect fluid [40]. Combining these two facts, we can intuitively understand that the black hole Skyrme hair with a sixth-order derivative term — possessing the properties of a perfect fluid — cannot withstand the gravitational attraction: the black hole Skyrme hair collapses.

It is not clear at this point if all higher-order derivative terms higher than fourth order will be unable to stabilize black hole Skyrme hair. Higher-order terms may not yield a theory with the properties of a perfect fluid. We will leave this question for future studies. We conjecture that the sixth-order derivative term, due to its properties of a perfect fluid, is the only higher-order derivative term leading to a second-order radial equation of motion which cannot stabilize the black hole Skyrme hair. The proof thereof awaits to be found.

**Note added.** While this manuscript was under preparation we were informed<sup>2</sup> that a paper with similar results was about to appear on the arXiv [41].

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